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Simplified formulae for solitons in media with slowly varying inhomogeneity

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Simplified formulae are obtained for the motion of solitons in nonuniform media, and the results are illustrated for the case of a sinusoidally varying medium.

Recently, Chen and Liu[1] presented an approximate theory of soliton propagation in media with slowly varying inhomogeneity, within the context of the adiabatic approximation. They considered the one-dimensional nonlinear Schrödinger equation for the amplitude q including an inhomogeneity term δn, namely,

$$\left(i\frac{d}{dt}+\frac{d^2}{dx^2}+2|q|^2-\delta n(x)\right)q(x,t)=0\ , \tag{1}$$

where t is time and x is position. They showed that to second order in the inhomogeneity, the adiabatic approximation leads to the solution

$$\begin{aligned}
&q=u(y,t)\exp[-iyp(t)-ir(t)],\quad y=x+m(t),\\
&p(t)\equiv\int_0^t f(s)ds,\quad m(t)\equiv 2\int_0^t p(s)\,ds,\\
&r(t)\equiv\int_0^t\left[e(s)-p^2(s)-f(s)m(s)\right]ds\ ,\\
&f(s)\equiv\frac{d}{dx}\,\delta n(x)\Big|_{x=\bar{x}(s)}\ ,\quad e(s)\equiv\delta n(\bar{x}(s))-\bar{x}(s)f(s),
\end{aligned} \tag{2}$$

where u satisfies the nonlinear Schrödinger equation in the absence of inhomogeneity, i.e.,

$$\left(i\frac{d}{dt}+\frac{d^2}{dy^2}+2|u|^2\right)u(y,t)=0 \tag{3}$$

and $\bar{x}$, the classical approximation for the pulse center, satisfies the equation

$$\int_{x_0}^{\bar{x}(t)}\frac{dx}{2[Q-\delta n(x)]^{1/2}}=t-t_0\ , \tag{4}$$

where Q is an appropriate constant. [We have corrected certain errors in Ref. 1, with respect to signs in the expressions for y and $r(t)$ in Eqs. (2), and a factor of 2 in the integral in Eq. (4)]. Combining Eq. (2) with the known solutions of Eq. (3), one obtains the single-soliton solution

$$\begin{aligned}
q(x,t)&=2\eta\,\mathrm{sech}\{2\eta[x+4\xi t+m(t)]\}\\
&\quad\times\exp(-i\{[2\xi+p(t)][x+m(t)]-4(\xi^2-\eta^2)t+r(t)\}).
\end{aligned} \tag{5}$$

If, for simplicity, henceforth we consider δn's chosen such that $\delta n(x_0)=0$, then we may identify

$$-2\xi=\sqrt{Q}\ . \tag{6}$$

Inspection of Eq. (5) indicates that in the presence of inhomogeneity the soliton acquires a time-dependent effective group velocity v, frequency ω, and propagation constant k of the form

$$\begin{aligned}
&v=v_0-m(t)/t;\quad k=k_0-p(t)\ ,\\
&\omega=\omega_0+[r(t)+m(t)p(t)-k_0m(t)]/t\ ,
\end{aligned} \tag{7}$$

where $v_0=-4\xi$, $k_0=-2\xi$, and $\omega_0=-4(\xi^2-\eta^2)$ are the values corresponding to zero inhomogeneity.

At first glance, it appears that to obtain the quantities in Eq. (7), one must first evaluate the integral in Eq. (4), invert the result to obtain $\bar{x}=\bar{x}(t)$, and use the latter to determine $f=f(t)$ and $e=e(t)$. Finally, one would perform the series of repeated integrations over t involving f and e, as indicated in Eq. (2). In this note, we indicate how explicit expressions may be obtained for r, m, and t (and, consequently, v, k, and ω) which do *not* involve any integrations beyond an appropriate equivalent of Eq. (4). The required results follow by changing the integration variable in Eq. (2) from t to $\bar{x}$; a transformation commonly utilized, for example, when calculating particle orbits in electric and magnetic fields. When this is done, one finds directly that [for $\delta n(x_0)=0$]

$$\begin{aligned}
&p(t)=\tfrac{1}{2}v_0-R(t),\quad m(t)=v_0t-\bar{x}(t),\\
&r(t)=-W(t)+v_0tR(t),
\end{aligned} \tag{8a}$$

$$R(t)\equiv\{k_0^2-\delta n[\bar{x}(t)]\}^{1/2}\ ,\quad W(t)\equiv\int_0^{\bar{x}(t)}[k_0^2-\delta n(x)]^{1/2}dx. \tag{8b}$$

To obtain the relation between $\bar{x}$ and t, one need not carry out the integration in Eq. (4), but may instead utilize

$$\frac{\partial W}{\partial k_0^2} = t - t_0 . \tag{9}$$

Since W is a functional of $\bar{x}(t)$, Eq. (9) may also be inverted to obtain $\bar{x} = \bar{x}(t)$. The quantities v, k, and ω now take the simple form

$$v = \bar{x}(t)/t , \quad k = R(\bar{t}), \quad \omega = \omega_0 + kv - W(t)/t . \tag{10}$$

Note especially that the soliton group velocity v involves no further integrations beyond either the one in Eq. (4) or (8b). Moreover, we have derived the very simple and perhaps not unexpected result, that within the adiabatic approximation, the soliton envelope becomes a function of $x - \bar{x}(t)$. When $\delta n \to 0$, $\bar{x} \to v_0 t$ and the envelope depends on $x - v_0 t$ as expected.

As an illustration of the present formulation, we consider the case of a periodic inhomogeneity of the form

$$\delta n(x) = C \sin^2 \alpha x . \tag{11}$$

A variation of this type could apply, for example, to the self-focusing of optical pulses in media with a sinusoidally varying refractive index.[2] Then, Eq. (8b) yields ($t_0 = 0$)

$$\begin{aligned} W(t) &= \alpha^{-1} k_0 E[\sigma, \alpha\bar{x}(t)]; \quad \sigma^2 \equiv C/k_0^2; \\ t &= \frac{\partial W}{\partial k_0^2} = F[\sigma, \alpha\bar{x}(t)](2\alpha k_0)^{-1} , \end{aligned} \tag{12}$$

where F and E are incomplete elliptic integrals of the first and second kind,[3] respectively. One obtains v directly by inverting Eq. (12) to yield $\bar{x} = \bar{x}(t)$, which in the present case requires, in general, numerical computation; q and ω take the form

$$\begin{aligned} k &= k_0[1 - \sigma^2 \sin^2 \alpha\bar{x}(t)]^{1/2} , \\ \omega &= \omega_0 + kv - k_0 E(\sigma, \alpha\bar{x})(\alpha t)^{-1} . \end{aligned} \tag{13}$$

For the special case $\sigma^2 = 1$, the integrals in Eq. (12) are elementary[4] and one is able to express $\bar{x}$ explicitly as a function of t,

$$\begin{aligned} \bar{x} &= \alpha^{-1} \sin^{-1} \gamma(t), \quad \gamma(t) = (e^s - 1)/(e^s + 1), \\ s &= 2\alpha v_0 t , \end{aligned} \tag{14}$$

whence

$$\begin{aligned} v &= \sin^{-1}\gamma(t)/\alpha t , \quad k = 2k_0 e^{s/2}/(e^s + 1), \\ \omega &= \omega_0 + kv - k_0 \gamma(t)/\alpha t . \end{aligned} \tag{15}$$

Note that it is also possible to obtain closed form expressions for v, k, and ω for $\sigma^2 \neq 1$ by approximating E and F with an appropriate series expansion. However, it is clear from Eq. (10) that one must proceed beyond the linear approximation for $\bar{x}(\bar{x} \approx v_0 t)$ to obtain a change in the soliton group velocity.

In conclusion, we have obtained simplified formulae for the effective soliton group velocity, frequency, and propagation constant in media characterized by slowly varying inhomogeneity, and have illustrated the results for the case of a $\sin^2 \alpha x$ variation.

[1] H. H. Chen and C. S. Liu, Phys. Fluids 21, 377 (1978).
[2] See, e.g., G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974), Sec. 16.4.
[3] S. M. Selby, editor, *CRC Standard Mathematical Tables* (Chemical Rubber Company, Boca Raton, Fla., 1968), 16th ed., p. 497.
[4] L. M. Milne Thomson, in *Handbook of Mathematicl Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1965), p. 594, Eqs. (17.4.21) and (17.4.25).

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